University of California, Berkeley
Physics 110A Spring 2003 Section 2 (*Strovink*)

Solution Set 6 (Compiled by Uday Varadarajan)

1. Using cyclindrical symmetry, we know that the magnetic field is always in the $\hat{\boldsymbol{\phi}}$ direction and independent of z. Thus, we apply Ampere's Law to circles centered at the origin in the $z = 0$ plane of successively larger radii,

$$\oint_{<a} \vec{B}_{<a} \cdot d\vec{l} = 2\pi s B_{<a}(s) = \mu_0 I_e = \mu_0 I \left(\frac{s^2}{a^2}\right) \Rightarrow B_{<a}(s) = \frac{\mu_0 I s}{2\pi a^2}, \tag{1}$$

$$\oint_{a<s<b} \vec{B}_{a<s<b} \cdot d\vec{l} = 2\pi s B_{a<s<b}(s) = \mu_0 I_e = \mu_0 I \Rightarrow B_{a<s<b}(s) = \frac{\mu_0 I}{2\pi}, \tag{2}$$

$$\oint_{b<s<c} \vec{B}_{b<s<c} \cdot d\vec{l} = 2\pi s B_{b<s<c}(s) = \mu_0 I_e = \mu_0 I \left(\frac{s^2 - b^2}{c^2 - b^2}\right) \Rightarrow B_{<a}(s) = \frac{\mu_0 I (s^2 - b^2)}{2\pi s (c^2 - b^2)}, \tag{3}$$

$$\oint_{>c} \vec{B}_{>c} \cdot d\vec{l} = 2\pi s B_{>c}(s) = \mu_0 I_e = 0 \Rightarrow B_{>c}(s) = 0. \tag{4}$$

$$\tag{5}$$

2. (a) To compute the magnetic field, we will use the result of Example 5.10 of Griffiths. It is shown there that *the magnetic field of the toroid is circumferential at all points, both inside and outside the coil.* The proof goes as follows. The Biot-Savart Law, tells us that the magnetic field at $\vec{r}$ due to some element of current at $\vec{r}'$ is proportional to $\frac{\vec{I} \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}$. Now, suppose $\vec{r}$ is in the xz plane, $(s, \phi = 0, z)$ and consider some element of current located at some some point (s', ϕ', z'). By assumption, $\vec{I}$ does not have a ϕ component $\vec{I} = (I_s, 0, I_z)$. Note that a current element at the point $\vec{r}'' = (s', -\phi', z')$ is the same distance away from $\vec{r}$ $(|\vec{r} - \vec{r}'| = |\vec{r} - \vec{r}''|)$, and by symmetry, has the same value of $\vec{I}$, and thereby makes a contribution to $\vec{B}(\vec{r})$ of the same magnitude. Thus, the sum of their contributions is proportional to,

$$d\vec{B} \propto \frac{\vec{I} \times (\vec{r} - \vec{r}' + \vec{r} - \vec{r}'')}{|\vec{r} - \vec{r}'|^3} \propto (I_s, 0, I_z) \times (2s - 2s', 0, 2z - 2z') \propto \hat{\boldsymbol{\phi}}, \tag{6}$$

where we used the fact that the cross product of two vector, neither of which have a ϕ component must lie in the $\hat{\boldsymbol{\phi}}$ direction. Thus, the Magnetic field is always circumferential.

With this knowledge, and the rotational symmetry of the toroid, we can just apply Ampere's law to a circle of radius s about the axis of the toroid,

$$B 2\pi s = \mu_0 I_{enc} \tag{7}$$

to get,

$$\vec{B}(\vec{r}) = \begin{cases} \frac{\mu_0 N I}{2\pi s} \hat{\boldsymbol{\phi}}, & \text{inside the toroid,} \\ 0, & \text{outside the toroid.} \end{cases} \tag{8}$$

(b) The magnetic flux linked by a given turn is just the integral of the magnetic field over the rectangular cross-section of the toroid, so

$$\Phi_B = N \int \vec{B} \cdot d\vec{a} = N \int_0^h dz \int_a^b ds \frac{\mu_0 N I}{2\pi s} = \frac{\mu_0 I}{2\pi} N^2 h \ln(b/a) \Rightarrow L = \Phi_B / I = \frac{\mu_0}{2\pi} N^2 h \ln(b/a). \tag{9}$$

3. (a) We imagine the Earth to be an ideal magnetic dipole with magnetic moment $\vec{m} = (0, 0, -m)$ located at its core. We are given the fact that the magnetic field of this dipole at the point $\vec{r} = (R/\sqrt{2}, 0, R/\sqrt{2})$ (where $R = 6.37 \times 10^6 m$ is the radius of the earth) has a magnitude of half a gauss. The magnitude of the magnetic field at $\vec{r}$ due to an ideal dipole at the origin is given by Griffiths Eqn. 5.87,

$$|\vec{B}(\vec{r})| = \frac{\mu_0}{4\pi r^3} |3(\vec{m} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \vec{m}| = \frac{\mu_0}{4\pi R^3} |(-3m/2)(\hat{\mathbf{x}} + \hat{\mathbf{z}}) + +m\hat{\mathbf{z}}| = \frac{\sqrt{10}\mu_0 m}{8\pi R^3}. \tag{10}$$

Solving for m, we find,

$$m = \frac{8\pi R^3 |\vec{B}|}{\sqrt{10}\mu_0} = \frac{8\pi(6.37\times 10^6 m)^3(0.5\times 10^{-4}T)}{\sqrt{10}4\pi\times 10^{-7}T\cdot m/A} = 8\times 10^{22} A\cdot m^2 \tag{11}$$

(b) The total magnetic moment of an iron ball of magnetization M is just,

$$m = M\frac{4}{3}\pi r^3 \Rightarrow r = \left(\frac{3m\mu_0}{4\pi\mu_0 M}\right)^{1/3} = \left(\frac{38\times 10^{22}A\cdot m^2 4\pi\times 10^{-7}T\cdot m/A}{4\pi(2T)}\right)^{1/3} = 2.31\times 10^5 m = 231km. \tag{12}$$

4. (a) $\nabla\cdot\vec{B} = k_0(\frac{\partial y}{\partial x} + \frac{\partial x}{\partial y}) = 0$, and $\nabla\times\vec{B} = k_0(0,0,\frac{\partial y}{\partial y} - \frac{\partial x}{\partial x}) = 0$.

(b) The Lorentz force law tells us,

$$\vec{F} = q(\vec{v}\times\vec{B}) = qk_0(-\dot{z}x\hat{\mathbf{x}} + \dot{z}y\hat{\mathbf{y}} + (\dot{x}x - \dot{y}y)\hat{\mathbf{z}}), \tag{13}$$

In particular, we see that if $qk_0 > 0$ and the $\dot{z} > 0$ (the particle is moving up through the magnet), the force in the x direction always opposes the motion and is proportional to x and $\dot{z}$. Thus, the force is analogous to that of a spring with a spring constant given by $\dot{z}$. Now, as the force in the z direction depends on the quantity $\dot{x}x$, we see that if the particle is moving away from $x = 0$, this spring constant always increases. Thus, the particle will tend to focus towards the origin in the x axis. More quantitatively, if we set $y = 0$ in the above, we get the differential equations,

$$m\ddot{x} = -\dot{z}x, m\ddot{z} = \dot{x}x \Rightarrow m\frac{d}{dt}(\dot{z} - \frac{1}{2}x^2) = 0 \Rightarrow m\ddot{z} = -\frac{1}{2}x^3 - \dot{z}_0 x. \tag{14}$$

Thus, it is clear that this gives rise to a force that focuses the x-direction. Now, since the sign of the force is along the direction of motion on the y-axis, the opposite is true for the y direction.

(c) Note that we have $\vec{B} = \mu_0\vec{H}$ in the region in which the $\vec{B}$ field is given as it is free of magnetic materials. Outside this region and inside the iron yolk, we expect that $\vec{H}$ is vanishingly small. Now, we consider the boundary conditions for the $\mathcal{H}$ field,

$$\vec{H}^{\|}_{above} - \vec{H}^{\|}_{below} = \vec{K}_f\times\hat{\mathbf{n}} \tag{15}$$

Since $\vec{H}^{\|}_{above}$ (the magnetic field outside the magnet and in the iron yoke) vanishes for the four surfaces, we have,

$$-\vec{H}^{\|}_{below} = -\mu_0\vec{B}^{\|}_{below} = \vec{K}_f\times\hat{\mathbf{n}} \tag{16}$$

We are given that $\vec{B} = k_0(\hat{\mathbf{x}}y + \hat{\mathbf{y}}x)$, so we find,

$$\vec{B}^{\|}(x = b, |y| < b) = k_0\hat{\mathbf{y}}b = \frac{1}{\mu_0}\hat{\mathbf{x}}\times K_f \Rightarrow K_f = -k_0\mu_0 b\hat{\mathbf{z}}, \tag{17}$$

$$\vec{B}^{\|}(x = -b, |y| < b) = -k_0\hat{\mathbf{y}}b = -\frac{1}{\mu_0}\hat{\mathbf{x}}\times K_f \Rightarrow K_f = -k_0\mu_0 b\hat{\mathbf{z}}, \tag{18}$$

$$\vec{B}^{\|}(|x| < b, y = b) = k_0\hat{\mathbf{x}}b = \frac{1}{\mu_0}\hat{\mathbf{y}}\times K_f \Rightarrow K_f = k_0\mu_0 b\hat{\mathbf{z}}, \tag{19}$$

$$\vec{B}^{\|}(|x| < b, y = -b) = -k_0\hat{\mathbf{x}}b = -\frac{1}{\mu_0}\hat{\mathbf{y}}\times K_f \Rightarrow K_f = k_0\mu_0 b\hat{\mathbf{z}}. \tag{20}$$

5. We note that if we just make the substitution, $\frac{\vec{B}}{\mu_0}\to\vec{A}$ and $\vec{J}\to\vec{B}$ into the mathematical relations we are given, then they state that

$$\nabla\times\vec{A} = \vec{B} \Leftrightarrow 4\pi\vec{A}(\vec{r}) = \int d\tau' \frac{\vec{B}(\vec{r}')\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}, \tag{21}$$

as long as $\vec{B}$ vanishes at ∞.

6. and

7. (a) Just plugging into the relations we derived in the last homework for a particle in a magnetic field with initial velocity $\vec{v} = \frac{p_0}{m}\hat{\mathbf{y}}$, we find

$$x(t) = \frac{p_0}{m\omega}\left(1 - \cos(\omega t)\right) = \frac{p_0}{eB_0}\left(1 - \cos(\omega t)\right), \tag{22}$$

$$y(t) = \frac{p_0}{m\omega}\sin(\omega t) = \frac{p_0}{eB_0}\sin(\omega t), \tag{23}$$

and the corresponding velocities are,

$$v_x(t) = (p_0/m)\sin(\omega t), \tag{24}$$

$$v_y(t) = (p_0/m)\cos(\omega t) \tag{25}$$

where $\omega = eB_0/m$.

(b) We'll do this generally. First, we note that for any point on the curve,

$$\vec{A}(x(t), y(t)) = \frac{B_0}{2}\left(-s(t)\sin\phi(t)\hat{\mathbf{x}} + s(t)\cos\phi(t)\hat{\mathbf{y}}\right) = \frac{B_0}{2}(-y(t)\hat{\mathbf{x}} + x(t)\hat{\mathbf{y}}). \tag{26}$$

Then, we see that,

$$\vec{P} = \vec{p} + e\vec{A} = \left(-\frac{y(t)B_0}{2} + mv_x(t)\right)\hat{\mathbf{x}} + \left(\frac{x(t)B_0}{2} + mv_y(t)\right)\hat{\mathbf{y}} = \frac{p_0}{2}\sin\omega t\hat{\mathbf{x}} + \frac{p_0}{2}(1 + \cos\omega t)\hat{\mathbf{y}} \tag{27}$$

Now, we find that,

$$\vec{r} \times \vec{P} = (x(t)P_y(t) - P_x(t)y(t))\hat{\mathbf{z}} = \frac{p_0^2}{eB_0}\left((1 - \cos(\omega t))(1 + \cos\omega t) - \sin^2\omega t\right)\hat{\mathbf{z}} = 0. \tag{28}$$

(c) Outside the region $s < s_0$, we are told to assume that $\vec{A}$ becomes rapidly negligible. Thus, when the particle is released, we see that $\vec{r}$ and $\vec{p}$ point along the $\pm\hat{\mathbf{x}}$ direction and,

$$\vec{r} \times (\vec{p} + e\vec{A}) = 0 \tag{29}$$

Now, from cylindrical symmetry and the fact that the motion occurs on the $z = 0$ plane so the above equation only has a non-trivial z component, we expect this quantity to be conserved along the path. Further, by symmetry, we expect that $\vec{A}$ is purely circumferential, so only $A_\phi(s)$ is non-zero and is only a function of radius in the $z = 0$ plane. In particular, we have that,

$$\oint \vec{A} \cdot d\vec{l} = 2\pi s A_\phi(s) = \Phi_B(s) \Rightarrow A_\phi(s) = \frac{\Phi_B(s)}{2\pi s}, \tag{30}$$

where $\Phi_B(s)$ is the magnetic flux through a disk of radius s about the origin. Further, note that we don't expect that the magnetic field will change the magnitude of the momentum of the particle, only its direction (as it cannot do any work), so we have $p(\vec{r}) = p_0$. Now, we can choose our origin anywhere we want it to be along the x-axis, and still find that the total canonical angular momentum about that origin vanishes, just as above. So choose it to be the point of entry into the magnet of the particle. Let $\theta/2$ be the angle between the x-axis and the vector $\vec{r}_e$, which is the point of exit. As the magnetic field and the vector potential rapidly decrease to zero outside the magnet, we can safely assume that $\vec{p}_e$ points normally outwards from the magnet, and so makes an angle of θ w.r.t to the x-axis. Since $\vec{A}$ is always circumferential, it is at an angle of $\theta - \pi/2$ w.r.t. the x-axis, and so conservation of canonical angular momentum gives us,

$$\vec{r}_e \times \vec{p}(\vec{r}_e) = -p_0 r_e \sin(\theta - \theta/2)\hat{\mathbf{z}} = -e\vec{r}_e \times \vec{A}(\vec{r}_e) = -er_e A(\vec{r}_e)\sin(\theta/2 - (\theta - \pi/2)) \tag{31}$$

$$\Rightarrow \tan(\theta/2) = \frac{eA(\vec{r}_e)}{p_e} = \frac{eA(s_0)}{p_0}, \tag{32}$$

Now, since we know that essentially all the magnetic flux is enclosed within the region $s < s_0$, and using the relation to magnetic flux, we have,

$$\theta = 2\tan^{-1}\left(\frac{e\Phi_B(s_0)}{2\pi s_0 p_0}\right). \tag{33}$$

This is the angle measured from the center of the magnet at which we expect to see the particle emerging from.

8. We calculate this flux by using Stoke's theorem to write,

$$\Phi_B = \int \vec{B}\cdot d\vec{a} = \int (\nabla \times \vec{A})\cdot d\vec{a} = \oint \vec{A}\cdot d\vec{l}. \tag{34}$$

Now, the vector potential for a magnetic dipole at the origin is given in Griffiths 6.10, with $\vec{m} = m\hat{\mathbf{z}}$,

$$\Phi_B = \oint \vec{A}\cdot d\vec{l} = \frac{\mu_0}{4\pi}\oint \left(\frac{m\hat{\mathbf{z}}\times\hat{\mathbf{r}}}{r^2}\right)\cdot d\vec{l} = \frac{\mu_0 m}{4\pi}\int_0^{2\pi}\frac{\sin\theta\hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}}d\phi}{b^2+z^2} = \frac{\mu_0 m b}{2(b^2+z^2)^{3/2}}. \tag{35}$$